DIFFRACTION OF (SHORT) SOUND WAVES AT AN OBSTACLE (CYLINDER, SPHERE, CONE AND PLANE)

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In the theory of sound, the solution of different problems of diffraction of sound waves at an obstacle (plane, lattice, slot, cylinder, sphere, ellipsoid, etc.) are studied. In this paper, the author applies the Poincaré [1] method, and presents a new solution to the problem.

1. Statement of the problem. From the theory of sound, it is known that in the case of harmonic motion with a time factor $e^{i\sigma t}$ the velocity potential of the sound waves satisfies the Poisson equation

$$\Delta\Phi + k^2\Phi = 0 \qquad \left(k = \frac{\sigma}{c} = \frac{2\pi}{\lambda}\right) \tag{1.1}$$

and the boundary condition at the surface of the object

$$\partial\Phi / \partial n = 0 \tag{1.2}$$

Here c is the speed of sound, λ is the wavelength, and n is the interior normal to the surface Σ. A solution of Equation (1.1) is sought in the form

$$\Phi = \frac{e^{-ikr'}}{r'} + f \tag{1.3}$$

Here, the first term represents the velocity potential of the spherical waves emanating from the source E, r' is the distance of the studied point M from the source E, and the second term is a function of the disturbance caused by the waves reflected from the surface Σ. Function f is determined from the relation

$$f = \iint_{\Sigma} \rho \frac{e^{-ikR}}{R} d\sigma \tag{1.4}$$

where ρ is the density of the potential sources of the reflected waves at the surface Σ, and R is the distance between an arbitrary point of the surface Σ and the studied point M where the velocity potential Φ is determined (Fig. 1).

The theory of the Newtonian potential for a straight acoustic layer is applied here for the determination of ρ at the surface Σ. It is known that the normal component velocity, when passing through the surface Σ, undergoes a discontinuity [2], i.e.

$$\frac{\partial V_e}{\partial n} - \frac{\partial V_0}{\partial n} = 2\pi\rho \qquad (1.5)$$

where $\partial V_e/\partial n$ is the exterior normal derivative of the potential of the acoustic layer, and $\partial V_0/\partial n$ is the direct value of the normal derivative at a point of the surface Σ.

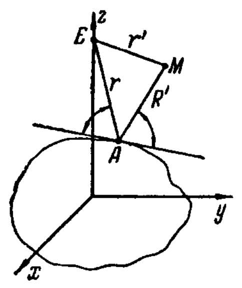

Fig. 1.

For the problem studied, condition (1.5) has the following form:

$$\rho = \frac{1+ikr}{2\pi r^2} e^{-ikr} \cos\psi + \frac{1}{2\pi} \iint_{\Sigma} \rho \frac{1+ikR'}{R'^2} \frac{\partial R'}{\partial n} e^{-ikR'} d\sigma \qquad (1.6)$$

Here r is the distance of the source E from a fixed point of the surface Σ, and R' is the distance between the fixed and the variable point of the surface Σ. The symbol ψ denotes the angle between r and the interior normal to the surface Σ at the point where r is determined.

Thus, in order to determine the function f, it is above all necessary to solve the integral equation (1.6) for different obstacle surfaces Σ.

2. Solution of the integral equation for a cylindrical obstacle. Assume that there is an infinite cylindrical tube with a cross-sectional radius a. Outside the cylindrical tube, at a distance l from the axis of the cylinder, there exists a sound source E with a constant intensity I_0.

Two planes containing E and tangent to the surface of the cylinder are formed. The part of the surface closer to the source E will be denoted by (C), the other part of the surface Σ is denoted by $(\Sigma - C)$ (Fig. 2).

We shall prove the following statement. With an accuracy up to terms of the order $1/ka$ in comparison to unity, the density ρ, which satisfies Equation (1.6), is given by the formula

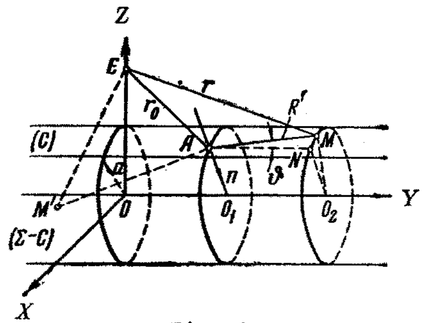

Fig. 2.

$$\rho = \frac{1 + ikr}{2\pi r^2} e^{-ikr} \cos\psi \quad \text{on } (C) \qquad (2.1)$$

$$\rho = 0 \quad \text{on } (\Sigma - C) \qquad (2.2)$$

For the derivation of Formula (2.1), we take a point A at the surface (C). For this point, we find the second term of the left part of Formula (1.6).

Using a change of variables in the integrand of (1.6) and after a simple transformation, the integral (1.6) reduces to the following form:

$$\iint\limits_{(\Sigma)} \rho \frac{1 + ikR'}{R'^2} \frac{\partial R'}{\partial n} e^{-inR'} d\sigma = \iint\limits_{(C)} \rho \frac{1 + ikR'}{R'^2} \frac{\partial R'}{\partial n} e^{-ikR'} d\sigma$$

$$= \iint\limits_{(C)} \frac{1 + ikr}{2\pi r^2} \frac{1 + ikR'}{R'^2} \cos\psi \frac{\partial R'}{\partial n} e^{-ik(R'+r')} d\sigma$$

$$= \iint\limits_{(C)} \frac{1 + ikr}{2\pi r^2} \frac{1 + ikR'}{\sqrt{4a^2 - R'^2 \sin^2\vartheta}} e^{-ik(R'+r')} \cos\psi \sin^2\vartheta \, dR' d\vartheta \qquad (2.3)$$

We shall show that this integral (2.3) is small as compared to ka, and can, therefore, be neglected. For this purpose, we use the Poincaré method, i.e. the following formula for the determination of an approximate value of the double integral:

$$\iint \eta(x, y) e^{ik\omega(x, y)} d\sigma = \eta(0, 0) \, e^{ik\omega(0, 0)} \frac{2\pi}{k\sqrt{\nu\mu}} \exp\left(\pm \frac{\pi i}{2}\right) \qquad (2.4)$$

Here $\nu = \omega_x''$; $\mu = \omega_y''$; the plus sign is used when $\nu > 0$ and $\mu > 0$, and the minus sign is used when $\nu < 0$ and $\mu < 0$. The application of Formula (2.4) to (2.3) yields

$$\iint\limits_{(C)} \frac{1 + ikr}{2\pi r^2} \frac{1 + ikR'}{\sqrt{4a^2 - R'^2 \sin^2\vartheta}} e^{-ik(R'+r)} \cos\psi \sin^2\vartheta \, dR d\vartheta = fe^{-ikr_0} + O[(ka)^{-1}] \quad (|f| < 1)$$

By neglecting this integral in comparison to ka, we obtain from Formula (1.6) Formula (2.1). In order to prove the statement (2.2) at the surface $(\Sigma - C)$, we show that

$$\frac{1}{2\pi}\iint\limits_{(\Sigma)} \rho \frac{1+ikR'}{R'^2}\frac{\partial R'}{\partial n} e^{-ikR'} d\sigma = -\frac{1+ikr}{2\pi r^2} e^{-ikr}\cos\psi \qquad (2.5)$$

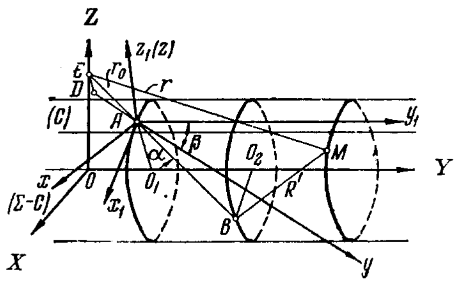

Fig. 3.

Let us choose a point B on the surface $(\Sigma - C)$ and connect it with point E by a straight line EB which intersects the surface of the cylinder Σ at point $A (A < C)$ (Fig. 3).

Let us construct confocal ellipsoids of revolution with foci at points E and B. Then one of them will necessarily be tangent to the surface of the cylinder. This point lies at the part of the surface $(\Sigma - C)$ where $\rho = 0$.

The equation of the surface of the cylinder in the coordinate system $Ax_1y_1z_1$ (axis $Ay_1 \parallel Oy$ lies on Σ, and axis $Ax_1 \perp y_1Az_1$ where Az_1 coincides with the exterior normal at point A of the surface Σ) has the form

$$x_1^2 + z_1^2 + 2az_1 = 0 \qquad (2.6)$$

Instead of $Ax_1y_1z_1$ let us take another system of coordinates $Axyz$ (axis Az coincides with the Az_1-axis, axis Ay coincides with the projection of the line EB onto the tangent plane at the point A of the surface Σ, and the axis Ax is perpendicular to the plane zAy).

Then the equation of the surface of the cylinder in the coordinate system $Axyz$ has the form

$$(x\sin\beta + y\cos\beta)^2 + z^2 + 2az = 0 \qquad (2.7)$$

where β is the angle between the axes Ay_1 and Ay.

Then the distances of the points E and B from the arbitrary variable point M on the surface Σ are given by

$$\begin{aligned} r^2 = EM^2 &= x^2 + (y + r_0\cos\alpha)^2 + (z - r_0\sin\alpha)^2 \\ R'^2 = BM^2 &= x^2 + (y - r_0\cos\alpha)^2 + (z + r_0\sin\alpha)^2 \end{aligned} \qquad (2.8)$$

where x, y, z are the coordinates of point M, $r_0 = EA$, $R_0 = AB$ and α is the angle between the line EA and the tangent plane at point A of the cylinder surface Σ.

By expanding r and R' into a Taylor series and taking into account Formula (2.7) we find $R' + r$ in the form

$$R' + r = R_0 + r_0 + \frac{x^2}{2}\left(\frac{1}{R_0} + \frac{1}{r_0}\right) + \frac{y^2}{2}\sin^2\alpha\left(\frac{1}{R_0} + \frac{1}{r_0}\right) + \cdots$$

Then, on the basis of Formulas (2.1), (2.4) and (2.8), the double integral in (1.6) is equal to

$$\frac{1}{2\pi}\iint\limits_{(\Sigma)} \rho\,\frac{1 + ikR'}{R'^2}\,\frac{\partial R'}{\partial n}\,e^{-inR'}\,d\sigma =$$

$$= \frac{1}{2\pi}\iint\limits_{(C)} \frac{1 + ikr}{2\pi r^2}\,\frac{1 + ikR'}{R'^2}\cos\psi\,\frac{\partial R'}{\partial n}\exp\left[-ik(R' + r)\right] d\sigma = -\frac{1}{2\pi}\cdot\frac{1 + ikr_0}{2\pi r_0^2}\cos\psi_0$$

$$\frac{1 + ikR_0}{R_0^2}\left(\frac{\partial R'}{\partial n}\right)_0 \frac{2\pi\exp\left[-ik(R_0 + r_0)\right]}{k(R_0^{-1} + r_0^{-1})\sin\alpha} = \frac{ki(R_0 + r_0)}{2\pi(R_0 + r_0)^2}(\cos\psi)_B\exp\left[-ik(R_0 + r_0)\right] \tag{2.9}$$

but

$$\left[-\frac{(\partial R'/\partial n)_0\cos\psi_0}{(\cos\psi)_B\sin\alpha}\right] = 1$$

Consequently, we have for the point B: $R_0 + r_0 = r$. After omitting the index B at $\cos\psi$, we have

$$\frac{1}{2\pi}\iint\limits_{(\Sigma)} \rho\,\frac{i + ikR'}{R'^2}\,\frac{\partial R'}{\partial n}\,e^{-ikR'}\,d\sigma = -\frac{ikr}{2\pi r^2}\,e^{-ikr}\cos\psi = -\frac{1 + ikr}{2\pi r^2}\,e^{-ikr}\cos\psi$$

Thus, the validity of Formula (2.5) is proven. Therefore, at the surface of the cylinder $(\Sigma - C)$, we have $\rho = 0$.

3. Determination of the function f for a cylindrical obstacle. Here, the following assumptions have been made:

Two tangent planes running from point E to the surface of the cylinder divide the space into three regions. Region I is enclosed between the tangent planes and the (C) part of the surface, region II is enclosed between the tangent planes and the $(\Sigma - C)$ part of the surface, region III is the remaining part of the space, (Fig. 4).

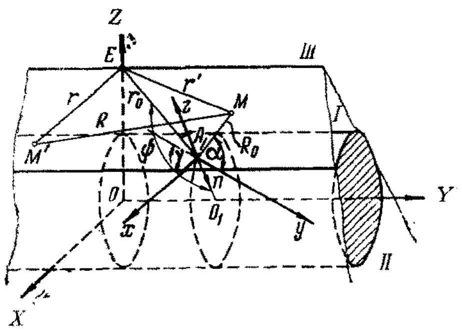

Fig. 4.

I. Assume that the studied point M lies in region I or in region III. Choose the origin A of a moving coordinate system $Axyz$. Construct confocal ellipsoids of revolution with foci at points E and M. Then one of them will be tangent to the surface

(C) and the other will touch the surface $(\Sigma - C)$. It is interesting to look at the case when point A lies on the surface (C), since $\rho = 0$ on $(\Sigma - C)$.

We draw two lines AE and AM from point A, which are equally inclined at an angle α to the tangent plane at point A of the surface Σ.

Then the plane EAM passes through the normal O_1A to the surface Σ. Let us use the coordinate system $Axyz$ (Fig. 4). An expansion of $R + r$ into a Taylor series yields

$$R + r = R_0 + r_0 + \frac{x^2}{2}\left[\frac{1}{R_0} + \frac{1}{r_0} + \frac{2\sin\alpha\sin^2\beta}{a}\right] + \\ + \frac{y^2}{2}\left[\sin^2\alpha\left(\frac{1}{R_0} + \frac{1}{r_0}\right) + \frac{2\sin\alpha\cos^2\beta}{a}\right] + \cdots \tag{3.1}$$

On the basis of Formulas (1.4), (2.1), (2.4) and (3.1), the function f is determined as follows:

$$f = \iint\limits_{(\Sigma)} \frac{1 + ikr}{2\pi r^2}\frac{\cos\psi}{R_0} e^{-ik(R_0+r_0)} \exp\left\{- ki\left[\frac{x^2}{2}\left(\frac{1}{R_0} + \frac{1}{r_0} + \frac{2\sin\alpha\sin^2\beta}{a}\right) + \right.\right. \\ \left.\left. + \frac{y^2}{2}\left\{\sin^2\alpha\left(\frac{1}{R_0} + \frac{1}{r_0}\right) + \frac{2\sin\alpha\cos^2\beta}{a}\right\}\right]\right\} d\sigma \\ = \frac{1 + ikr_0}{2\pi r_0^2}\frac{\cos\psi_0}{R_0} e^{-ik(R_0+r_0)} \frac{-2\pi i}{k}\left\{\left(\frac{1}{R_0} + \frac{1}{r_0} + \frac{2\sin\alpha\sin^2\beta}{a}\right) \times \right. \\ \left. \times\left[\sin^2\alpha\left(\frac{1}{R_0} + \frac{1}{r_0}\right) + \frac{2\sin\alpha\cos^2\beta}{a}\right]\right\}^{-1/2} = \frac{1 + ikr_0}{2\pi r_0}\frac{-2\pi i\cos\psi_0}{k\dfrac{R_0 + r_0}{a}} \times \\ \times e^{-ik(R_0+r_0)}\left\{\left(a + \frac{2R_0r_0\sin\alpha\sin^2\beta}{R_0 + r_0}\right)\left(a\sin^2\alpha + \frac{2R_0r_0\sin\alpha\cos^2\beta}{R_0 + r_0}\right)\right\}^{-1/2}$$

The problem is analysed with an accuracy up to terms of the order $1/ka$ as compared to unity, i.e. $1 \ll ka$. Since $a < r_0$, then $1 \ll ka < kr_0$. Thus, we have

$$\frac{1 + ikr_0}{2\pi r_0}\frac{-2\pi i}{k} \approx \frac{ikr_0}{2\pi r_0}\frac{-2\pi i}{k} = 1$$

Consequently

$$f = \cos\psi_0 e^{-ik(R_0+r_0)}\left\{\left(\frac{1}{R_0} + \frac{1}{r_0} + \frac{2\sin\alpha\sin^2\beta}{a}\right) \times \right. \\ \left. \times\left[\sin^2\alpha\left(\frac{1}{R_0} + \frac{1}{r_0}\right) + \frac{2\sin\alpha\cos^2\beta}{a}\right]\right\}^{-1/2} \tag{3.2}$$

Since $\cos\psi_0 = \cos(1/2\,\pi + \alpha) = \sin\alpha$ we obtain

$$f = -p\frac{e^{-ik(R_0+r_0)}}{R_0+r_0}\quad p = \frac{a\sqrt{\sin\alpha}}{\sqrt{\left(a+\frac{2R_0r_0\sin\alpha\sin^2\beta}{R_0+r_0}\right)\left(a\sin\alpha+\frac{2R_0r_0\cos^2\beta}{R_0+r_1}\right)}} \tag{3.3}$$

Using this, one can write Equation (1.1), taking into account the time factor $e^{i\sigma t}$ and the sound source intensity I_0, in the following form:

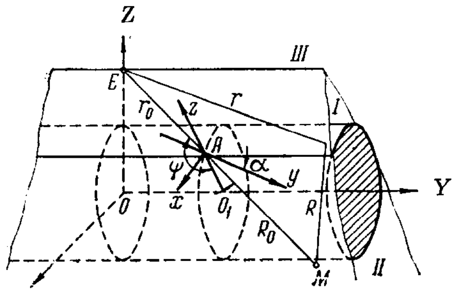

Fig. 5.

$$\Phi = I_0\frac{e^{i(\sigma t-kr')}}{r'} - I_0 p\frac{e^{i(\sigma t-kR_0-kr_0)}}{R_0+r_0} \tag{3.5}$$

II. Now we shall find function f, and consequently function Φ, when point M lies in region II (Fig. 5).

We shall show that region II is a sound shadow region.

The line EA intersects the surface Σ at two points. Let us take a point A that lies on the (C) part of the cylinder surface. Then relative to this moving point we construct a coordinate system $A\,xyz$, as shown in Fig. 5. An expansion of $R + r$ into a Taylor series yields then

$$R+r = R_0+r_0+\frac{x^2}{2}\left(\frac{1}{R_0}+\frac{1}{r_9}\right)+\frac{y^2}{2}\left(\frac{1}{R_0}+\frac{1}{r_0}\right)\sin^2\alpha+\cdots \tag{3.6}$$

On the basis of Formulas (1.4), (2.1), (2.4) and (3.6) function f becomes

$$f = \iint\limits_{(\Sigma)} p\frac{e^{-ikR}}{R}\,d\sigma = \iint\limits_{(C)} \frac{1+ikr}{2\pi r^2}\cos\psi\frac{e^{-ik(R+r)}\,d\sigma}{R}$$

$$= \frac{1+ikr_0}{2\pi r_0^2}\cos\psi_0\frac{e^{-ik(R_0+r_0)}}{R_0}\,\frac{-2\pi i}{k\,(R_0^{-1}+r_0^{-1})\sin\alpha} = -\frac{e^{-ik(R_0+r_0)}}{R_0+r_0} = -\frac{e^{-ikr'}}{r'} \tag{3.7}$$

Here, we assume that $R_0 + r_0 = r'$ for the studied point M. The solution of Equation (1.1) then becomes

$$\Phi = I_0\frac{e^{i(\sigma t-kr')}}{r'} - I_0\frac{e^{i(\sigma t-kr')}}{r'} = 0 \tag{3.8}$$

Thus, one can apply with an accuracy up to the order $O(1/ka)$ the approximate asymptotic formulas (3.5) and (3.8) for the computation of the velocity potential for short sound waves in the case of a cylindrical obstacle.

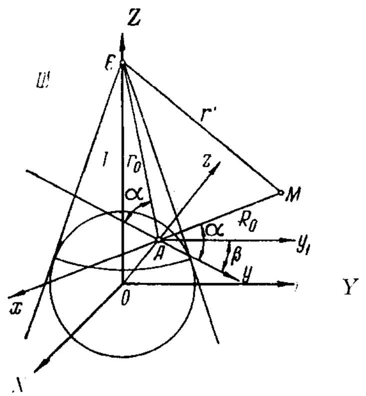

Fig. 6.

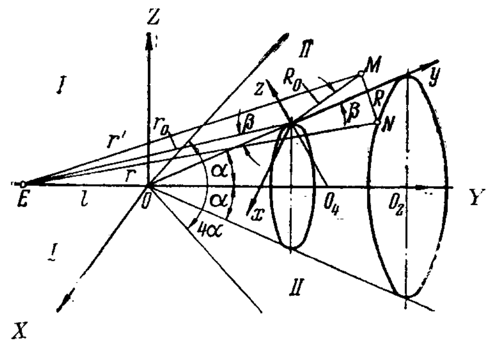

Fig. 7.

4. Solution of Equation (1.1) in the case of various obstacles. When applying Sections 2 and 3 of this paper to various obstacles, we obtain outside the shadow region

$$\Phi = I_0 \frac{e^{i(\sigma t - kr')}}{r'} - pI_0 \frac{e^{i(\sigma t - kR_0 - kr_0)}}{R_0 + r_0} \tag{4.1}$$

where p is determined as follows:

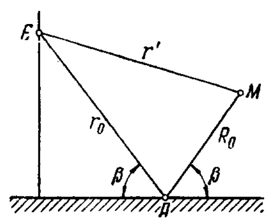

Fig. 8.

For a sphere (Fig. 6)

$$p = \frac{\sqrt{\sin\alpha}}{\sqrt{\left(\sin\alpha + \frac{2R_0r_0}{a(R_0+r_0)}\right)\left(1 + \frac{2R_0r_0}{a(R_0+r_0)}\sin\alpha\right)}} \tag{4.2}$$

For a cone (Fig. 7) in regions I and II, respectively

$$p = 0, \qquad p = \sin\beta\left(1 + \frac{2R_0r_0\sin\beta\cos^2\beta}{a\,(R_0+r_0)\sin\alpha}\right)^{-1/2} \tag{4.3}$$

For a plane (Fig. 8)

$$p = \sin\beta \tag{4.4}$$

In the shadow region $\Phi = 0$ for all above-mentioned obstacles.

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